SEVERAL REVERSE INEQUALITIES OF OPERATORS (Advanced Study of Applied Functional Analysis and Information Sciences)

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ABSTRACT. In this report, we show reverse inequalities to Araki's inequality and investigate the equivalence among reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities. Among others, we show that if A and B are positive operators on a Hilbert space H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$K(m, M, p)\|BAB\|^p \leq \|B^pA^pB^p\| \qquad \text{for all } 0 < p < 1,$$

where $K(m, M, p)$ is a generalized Kantorovich constant by Furuta.

1. INTRODUCTION

Let A and B be positive operators on a Hilbert space H. The equivalence among Cordes and Löwner-Heinz inequalities was discussed by many authors. In [8], Furuta showed that the Cordes inequality

$$\|A^pB^p\| \leq \|AB\|^p \qquad \text{for } 0 < p < 1 \tag{1}$$

is equivalent to the Löwner-Heinz inequality (cf.[14])

$$A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \quad \text{for } 0 < p < 1 \tag{2}$$

(cf. [5]). In [1], Araki showed a trace inequality which entails the following inequality:

$$\|B^pA^pB^p\| \leq \|BAB\|^p \quad \text{for } 0 < p < 1. \tag{3}$$

Moreover, it was shown in [6, 2] that the Cordes inequality (1) is equivalent to Araki's inequality (3).

On the other hand, Furuta [9] showed the following Kantorovich type inequalities: If A and B are positive operators with $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$A \geq B \geq 0 \quad \text{implies} \quad K(m, M, p)A^p \geq B^p \quad \text{for } p > 1, \tag{4}$$

where a generalized Kantorovich constant $K(m, M, p)$ [3, 7, 11] is defined as

(5)
$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p - m^p}{mM^p - Mm^p}\right)^p \quad \text{for all real numbers } p.$$

We here cite Furuta's textbook [10] as a pertinent reference to Kantorovich inequalities.

Also, Yamazaki [16] showed the following difference type reverse inequalities of the Löwner-Heinz inequality: If A and B are positive operators with $0 < mI \leq B \leq MI$ for some scalars $m < M$, then

$$A \geq B \geq 0 \quad \text{implies} \quad C(m, M, p) + A^p \geq B^p \quad \text{for } p > 1, \tag{6}$$

where the constant $C(m, M, p)$ [12, 16] is defined as

(7)
$$C(m,M,p) = (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M-m} \quad \text{for all real numbers } p.$$

In this report, we show reverse inequalities to Araki's inequality (3) and the Cordes inequality (1): If A and B are positive operators with $0 < mI \le A \le MI$ for some scalars $m < M$, then the following inequalities hold

$$K(m,M,p)\|BAB\|^p \le \|B^pA^pB^p\| \quad \text{for } 0 < p < 1, \tag{8}$$

$$K(m^2,M^2,p)^{1/2}\|AB\|^p \le \|A^pB^p\| \quad \text{for } 0 < p < 1, \tag{9}$$

respectively. We moreover show that reverse inequalities (4), (8) and (9) are mutually equivalent.

2. Main Results

First of all, we present our main theorem which is a reverse inequality to Araki's inequality (3).

Theorem 1. *If A and B are positive operators on H such that $0 < mI \le A \le MI$ for some scalars $m < M$, then for each $\alpha > 0$*

$$\|BAB\|^p \le \alpha\ \|B^pA^pB^p\| + \beta(m^p, M^p, \frac{1}{p}, \alpha)\|B\|^{2p} \qquad \text{for all } 0 < p < 1, \tag{10}$$

or equivalently

$$\|B^pA^pB^p\|^{\frac{1}{p}} \le \alpha\ \|BAB\| + \beta(m,M,p,\alpha)\|B\|^2 \qquad \text{for all } p > 1, \tag{11}$$

where

(12)
$$\beta(m,M,p,\alpha) = \begin{cases} \frac{p-1}{p}\left(\frac{M^p-m^p}{\alpha p(M-m)}\right)^{\frac{1}{p-1}} + \frac{\alpha(Mm^p - mM^p)}{M^p-m^p} & \text{if } \frac{M^p-m^p}{pM^{p-1}(M-m)} \le \alpha \le \frac{M^p-m^p}{pm^{p-1}(M-m)}, \\ (1-\alpha)M & \text{if } 0 < \alpha \le \frac{M^p-m^p}{pM^{p-1}(M-m)}, \\ (1-\alpha)m & \text{if } \alpha \ge \frac{M^p-m^p}{pm^{p-1}(M-m)}. \end{cases}$$

If we choose α satisfying $\beta(m, M, p, \alpha) = 0$ in Theorem 1, then we have the following ratio type reverse inequalities.

Corollary 2. *If A and B are positive operators on H such that $0 < mI \le A \le MI$ for some scalars $m < M$, then*

$$K(m,M,p)\|BAB\|^p \le \|B^pA^pB^p\| \qquad \text{for } 0 < p < 1, \tag{13}$$

or equivalently

$$\|BAB\|^p \le K(m,M,p)\|B^pA^pB^p\| \qquad \text{for } p > 1, \tag{14}$$

where $K(m, M, p)$ is defined as (5) in §1.

If we put $\alpha = 1$ in Theorem 1, then we have the following difference type reverse inequalities.

Corollary 3. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$\|BAB\|^p - \|B^pA^pB^p\| \leq -C(m, M, p)\|B\|^{2p} \qquad \text{for } 0 < p < 1, \tag{15}$$

or equivalently

$$\|B^pA^pB^p\|^{\frac{1}{p}} - \|BAB\| \leq -C(m^p, M^p, \frac{1}{p})\|B\|^2 \qquad \text{for } p > 1, \tag{16}$$

where $C(m, M, p)$ is defined as (7) in §1.

As special cases of Corollary 2 and Corollary 3, we have the following corollary.

Corollary 4. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$\|B^2A^2B^2\| \leq \frac{(M+m)^2}{4Mm}\|BAB\|^2. \tag{17}$$

$$\|B^2A^2B^2\|^{\frac{1}{2}} - \|BAB\| \leq \frac{(M-m)^2}{4(M+m)}\|B\|^2. \tag{18}$$

$$\frac{2\sqrt[4]{Mm}}{\sqrt{M}+\sqrt{m}}\|BAB\|^{\frac{1}{2}} \leq \|B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}\|. \tag{19}$$

$$\|BAB\|^{\frac{1}{2}} - \|B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{(\sqrt{M}-\sqrt{m})^2}{4(\sqrt{M}+\sqrt{m})}\|B\|. \tag{20}$$

Since $\|X^*X\| = \|X\|^2$ for an operator X, we obtain the following reverse inequality to the Cordes inequality by Corollary 2.

Theorem 5. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$K(m^2, M^2, p)^{\frac{1}{2}}\|AB\|^p \leq \|A^pB^p\| \qquad \text{for all } 0 < p < 1, \tag{21}$$

or equivalently

$$\|A^pB^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}}\ \|AB\|^p \quad \text{for all } p > 1. \tag{22}$$

In particular,

$$\sqrt{\frac{2\sqrt{Mm}}{M+m}}\|AB\|^{\frac{1}{2}} \leq \|A^{\frac{1}{2}}B^{\frac{1}{2}}\|. \tag{23}$$

and

$$\|A^2B^2\| \leq \frac{M^2+m^2}{2Mm}\|AB\|^2 \tag{24}$$

The equivalence among the reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities is now given as follows.

Theorem 6. *For a given $p > 1$, the following are mutually equivalent: For all $A, B \geq 0$ and $0 < mI \leq A \leq MI$*

(A) $\quad A \geq B \geq 0$ *implies* $K(m, M, p)A^p \geq B^p$.

(B) $\quad \|A^pB^p\| \leq K(m^2, M^2, p)^{1/2}\|AB\|^p$.

(C) $\quad \|B^pA^pB^p\| \leq K(m, M, p)\|BAB\|^p$.

(B') $\quad K(m^2, M^2, 1/p)^{1/2}\|AB\|^p \leq \|A^pB^p\|$.

(C') $\quad K(m, M, 1/p)\|BAB\|^p \leq \|B^pA^pB^p\|$.

3. Lemmas

We start with the following three lemmas before we give proofs of the results in §2.

Let A be a positive operator on a Hilbert space H and x a unit vector in H. Then it follows from Hölder-McCarthy inequality that

$$(Ax, x) \leq (A^px, x)^{\frac{1}{p}} \qquad \text{for all } p > 1. \tag{25}$$

By using the Mond-Pečarić method [12, 13], we have the following reverse inequality of (25) [15, 4]:

Lemma 7. *If A is a positive operator on H such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$*

$$(A^px, x)^{\frac{1}{p}} \leq \alpha(Ax, x) + \beta(m, M, p, \alpha) \qquad \textit{for all } p > 1 \tag{26}$$

holds for every unit vector $x \in H$, where $\beta(m, M, p, \alpha)$ is defined as (12) in Theorem 1.

Proof. For the sake of reader's convenience, we give a proof. Put $\beta = \beta(m, M, p, \alpha)$ and $f(t) = (at+b)^{\frac{1}{p}} - \alpha t$ for $a = \frac{M^p - m^p}{M-m}$ and $b = \frac{Mm^p - mM^p}{M-m}$, then we have $f'(t) = \frac{a}{p}(at+b)^{\frac{1}{p}-1} - \alpha$. It follows that the equation $f'(t) = 0$ has exactly one solution $t_0 = \frac{1}{a}(\frac{\alpha p}{a})^{\frac{p}{1-p}} - \frac{b}{a}$. If $m \leq t_0 \leq M$, then we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0)$ since $f''(t) = \frac{a^2(1-p)}{p^2}(at+b)^{\frac{1}{p}-2} < 0$ and the condition $m \leq t_0 \leq M$ is equivalent to the condition

$$\frac{M^p - m^p}{pM^{p-1}(M-m)} \leq \alpha \leq \frac{M^p - m^p}{pm^{p-1}(M-m)}.$$

If $M \leq t_0$, then $f(t)$ is increasing on $[m, M]$ and hence we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1-\alpha)M$ for $t_0 = M$. Similarly, we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1-\alpha)m$ for $t_0 = m$ if $t_0 \leq m$. Hence it follows that

$$(at+b)^{\frac{1}{p}} - \alpha t \leq \beta \qquad \text{for all } t \in [m, M].$$

Since t^p is convex for $p > 1$, it follows that $t^p \leq at + b$ for $t \in [m, M]$. By the spectral theorem, we have $A^p \leq aA + b$ and hence $(A^px, x) \leq a(Ax, x) + b$ for every unit vector $x \in H$. Therefore we have

$$\begin{aligned}(A^px, x)^{\frac{1}{p}} - \alpha(Ax, x) &\leq (a(Ax, x) + b)^{\frac{1}{p}} - \alpha(Ax, x) \\ &\leq \max_{m \leq t \leq M} f(t) = \beta(m, M, p, \alpha).\end{aligned}$$

□

By Lemma 7, we have the following estimates of both the difference and the ratio in the inequality (25).

Lemma 8. *If A is a positive operator on H such that $0 < mI \le A \le MI$ for some scalars $0 < m < M$, then for each $p > 1$*

$$(A^p x, x)^{\frac{1}{p}} \le K(m, M, p)^{\frac{1}{p}}(Ax, x) \tag{27}$$

and

$$(A^p x, x)^{\frac{1}{p}} - (Ax, x) \le -C(m^p, M^p, \frac{1}{p}) \tag{28}$$

hold for every unit vector $x \in H$, where $K(m, M, p)$ is defined as (5) in §1 and $C(m, M, p)$ is defined as (7) in §1.

Proof. If we choose α satisfying $\beta(m, M, p, \alpha) = 0$ in Lemma 7, then we have $\alpha = K(m, M, p)^{\frac{1}{p}}$. If we put $\alpha = 1$ in Lemma 7, then we have $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$. □

We remark that $K(m, M, 2)$ coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$ if $p = 2$.

We summarize some important properties of a generalized Kantorovich constant [3, 11].

Lemma 9. *Let $m < M$ be given. Then a generalized Kantorovich constant $K(m, M, p)$ has the following properties.*

(i) $K(m, M, p) = K(M, m, p)$ *for all* $p \in \mathbb{R}$.
(ii) $K(m, M, p) = K(m, M, 1-p)$ *for all* $p \in \mathbb{R}$.
(iii) $K(m, M, 0) = K(m, M, 1) = 1$ *for all* $p \in \mathbb{R}$.
(iv) $K(m, M, p)$ *is increasing for* $p > \frac{1}{2}$ *and decreasing for* $p < \frac{1}{2}$.
(v) $K(m^r, M^r, \frac{p}{r})^{\frac{1}{p}} = K(m^p, M^p, \frac{r}{p})^{-\frac{1}{r}}$ *for* $pr \ne 0$.

4. Proof of results

Based on Lemmas in the preceding section, we give proofs of the results mentioned in the second section.

Proof of Theorem 1.
For every unit vector $x \in H$, it follows that

$$\begin{aligned}
&((BAB)^p x, x)\\
\le\ & (BABx, x)^p \quad \text{by Hölder-McCarthy inequality and } 0 < p < 1\\
=\ & \left((A^p)^{\frac{1}{p}} \frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}\right)^p \|Bx\|^{2p}\\
\le\ & \left(\alpha(A^p \frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}) + \beta(m^p, M^p, \frac{1}{p}, \alpha)\right) \|Bx\|^{2p} \quad \text{by Lemma 7}\\
=\ & \alpha(A^p Bx, Bx)\|Bx\|^{2p-2} + \beta(m^p, M^p, \frac{1}{p}, \alpha)\|Bx\|^{2p}\\
=\ & \alpha\left(B^p A^p B^p \frac{B^{1-p}x}{\|B^{1-p}x\|}, \frac{B^{1-p}x}{\|B^{1-p}x\|}\right)\|Bx\|^{2p-2}\|B^{1-p}x\|^2 + \beta(m^p, M^p, \frac{1}{p}, \alpha)\|Bx\|^{2p}
\end{aligned}$$

and

$$\begin{aligned}\|Bx\|^{2p-2}\|B^{1-p}x\|^2 &= (B^2x,x)^{p-1}(B^{2-2p}x,x)\\ &\le (B^2x,x)^{p-1}(B^2x,x)^{1-p}=1 \quad \text{by } 0<1-p<1.\end{aligned}$$

By combining two inequalities above, we have

$$\begin{aligned}\|BAB\|^p &= \|(BAB)^p\|\\ &\le \alpha\|B^pA^pB^p\| + \beta(m^p,M^p,\frac{1}{p},\alpha)\|B\|^{2p}\end{aligned}$$

and hence we have the desired inequality (10).

Next, we show (10)$\Longrightarrow$(11). For $p>1$, since $0<\frac{1}{p}<1$, it follows from (10) that

$$\|BAB\|^{\frac{1}{p}} \le \alpha\ \|B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}}\| + \beta(m^{\frac{1}{p}},M^{\frac{1}{p}},p,\alpha)\|B\|^{\frac{2}{p}}.$$

By replacing A by A^p and B by B^p in the above inequality respectively, we have

$$\|B^pA^pB^p\|^{\frac{1}{p}} \le \alpha\ \|BAB\| + \beta(m,M,p,\alpha)\|B^p\|^{\frac{2}{p}},$$

and so we have the desired inequality (11). Similarly we can show (11)$\Longrightarrow$(10). Therefore (10) is equivalent to (11). □

Proof of Corollary 2.

For $p>1$, if we put $\beta(m,M,p,\alpha)=0$ in Theorem 1, then it follows that

$$\frac{p-1}{p}\left(\frac{M^p-m^p}{p(M-m)}\right)^{\frac{1}{p-1}} + \alpha^{\frac{p}{p-1}}\frac{(Mm^p-mM^p)}{M^p-m^p}=0$$

and hence

$$\alpha^{\frac{p}{p-1}} = -\frac{p-1}{p}\left(\frac{M^p-m^p}{p(M-m)}\right)^{\frac{1}{p-1}}\frac{M^p-m^p}{Mm^p-mM^p}.$$

Therefore, we have

$$\begin{aligned}\alpha^p &= \frac{M^p-m^p}{p(M-m)}\left(\frac{p-1}{p}\frac{M^p-m^p}{mM^p-Mm^p}\right)^{p-1}\\ &= K(m,M,p)\end{aligned}$$

and we obtain the desired inequality (14).

For $0<p<1$, since $1/p>1$, it follows from (14) that

$$\|BAB\|^{\frac{1}{p}} \le K(m,M,\frac{1}{p})\|B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}}\|.$$

By replacing A and B by A^p and B^p respectively, then we have

$$\|B^pA^pB^p\|^{\frac{1}{p}} \le K(m^p,M^p,\frac{1}{p})\|BAB\|.$$

Hence it follows from Lemma 9 that

$$\begin{aligned}\|B^pA^pB^p\| &\le K(m^p,M^p,\frac{1}{p})^p\|BAB\|^p\\ &\le K(m,M,p)^{-1}\|BAB\|^p,\end{aligned}$$

and we have the desired inequality (13). Similarly we have the implication (13)$\Longrightarrow$(14).
□

Proof of Corollary 3.
If we put $\alpha = 1$ in Theorem 1, then it follows that

$$\begin{aligned}\beta(m^p, M^p, \frac{1}{p}, 1) &= \frac{\frac{1}{p}-1}{\frac{1}{p}}\left(\frac{M-m}{\frac{1}{p}(M^p-m^p)}\right)^{\frac{1}{\frac{1}{p}-1}} + \frac{M^pm - m^pM}{M-m} \\ &= (1-p)\left(\frac{p(M-m)}{M^p-m^p}\right)^{\frac{p}{1-p}} + \frac{M^pm-m^pM}{M-m} \\ &= -C(m,M,p).\end{aligned}$$

Similarly it follows that $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$. Hence we have the equivalence (15) $\Longleftrightarrow$ (16) □

Proof of Corollary 4.
In Corollary 2 and 3, we have only to put $p = 2$ and $p = 1/2$. □

Proof of Theorem 5
By Corollary 2, it follows that

$$K(m,M,p)\|A^{\frac{1}{2}}B\|^{2p} \le \|A^{\frac{p}{2}}B^p\|^2.$$

By replacing A by A^2, we have

$$K(m^2,M^2,p)\|AB\|^{2p} \le \|A^pB^p\|^2.$$

Therefore we have (21). Similarly, we have the equivalence (21)$\Longleftrightarrow$(22). □

Proof of Theorem 6
The proof is divided into three parts, namely the equivalence $(A) \Longrightarrow (B) \Longrightarrow (C) \Longrightarrow (A)$, $(B) \Longleftrightarrow (B')$ and $(C) \Longleftrightarrow (C')$.
$(A) \Longrightarrow (B)$. It follows that

$$\begin{aligned}(A) &\Longleftrightarrow \|A^{-\frac{1}{2}}B^{\frac{1}{2}}\| \le 1 \to \|A^{-\frac{p}{2}}B^{\frac{p}{2}}\|^2 \le K(m,M,p) \\ &\Longleftrightarrow \|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \le 1 \to \|A^{\frac{p}{2}}B^{\frac{p}{2}}\|^2 \le K(M^{-1},m^{-1},p) = K(m,M,p) \\ &\Longleftrightarrow \|AB\| \le 1 \to \|A^pB^p\| \le K(m^2,M^2,p).\end{aligned}$$

If we put $B_1 = B/\|AB\|$, then it follows from $\|AB_1\| = 1$ that

$$\|A^pB_1^p\| \le K(m^2,M^2,p)^{\frac{1}{2}} \Longleftrightarrow \|A^pB^p\| \le K(m^2,M^2,p)^{\frac{1}{2}}\|AB\|^p.$$

$(B) \Longrightarrow (C)$. If we replace A by $A^{\frac{1}{2}}$ in (A), then it follows that

$$\|A^{\frac{p}{2}}B^p\| \le K(m,M,p)^{\frac{1}{2}}\|A^{\frac{1}{2}}B\|^p.$$

Square both sides, we have

$$\|B^pA^pB^p\| \le K(m,M,p)\|BAB\|^p.$$

$(C) \Longrightarrow (A)$. If we replace B by $B^{\frac{1}{2}}$ and A by A^{-1} in (C), then it follows that

$$\|B^{\frac{p}{2}}A^{-p}B^{\frac{p}{2}}\| \le K(M^{-1},m^{-1},p)\|B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\|^p.$$

By rearranging it, we have

$$\|A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}}\| \le K(m,M,p)\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\|^p.$$

Since $A \geq B \geq 0$, it follows from $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq 1$ that

$$\|A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}}\| \leq K(m, M, p)$$

and hence

$$B^p \leq K(m, M, p)A^p.$$

$(B) \iff (B')$: If we replace A and B by $A^{\frac{1}{p}}$ and $B^{\frac{1}{p}}$ in (B) respectively, then it follows that

$$\begin{aligned}
(B) &\iff \|AB\| \leq K(m^{\frac{2}{p}}, M^{\frac{2}{p}}, p)^{\frac{1}{2}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\|^p \\
&\iff \|AB\|^{\frac{1}{p}} \leq K(m^{\frac{2}{p}}, M^{\frac{2}{p}}, p)^{\frac{1}{2p}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \\
&\iff K(m^2, M^2, p)^{\frac{1}{2}}\|AB\|^{\frac{1}{p}} \leq \|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \quad \text{by Lemma 9} \\
&\iff (B')
\end{aligned}$$

Similarly we have $(C) \iff (C')$ and so the proof is complete. □

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